- To smooth out the irregular roughness means to see a clearer signal.

- In time series smoothing is used to sweep out the finer grains.

- For seasonal data, we might smooth out the seasonality so that we can identify the trend.

- Smoothing doesn't provide us with a model, but it can be a good first step in describing various components of the series.

- Common technique: Moving average method

Most sophisticated and advanced smoothing technique is

Exponential Smoothing Method

Exponential Smoothing Methods
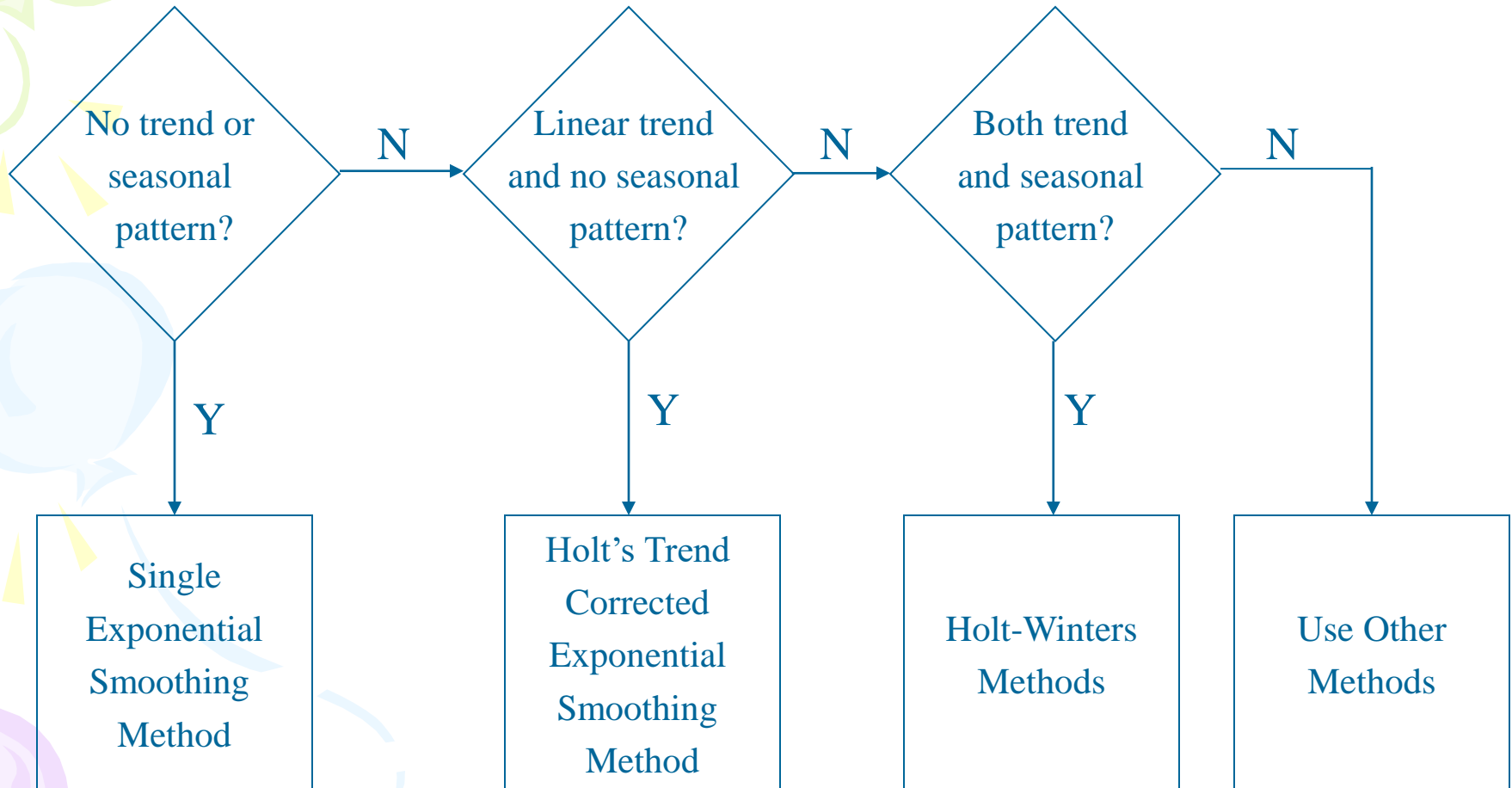
Motivation of Exponential Smoothing

- Simple moving average method assigns equal weights $(1/k)$ to all k data points.

- Arguably, recent observations provide more relevant information than do observations in the past.

- So we want a weighting scheme that assigns decreasing weights to the more distant observations.

Exponential Smoothing

- Exponential smoothing methods give larger weights to more recent observations, and the weights decrease exponentially as the observations become more distant.

- These methods are most effective when the parameters describing the time series are changing SLOWLY over time.

Data vs Methods

```
┌─────────────┐              ┌─────────────┐              ┌─────────────┐
│ No trend or │     N        │ Linear trend│     N        │ Both trend  │     N
│ seasonal    │ ─────────►   │ and no      │ ─────────►   │ and seasonal│ ─────────►
│ pattern?    │              │ seasonal    │              │ pattern?    │
└─────────────┘              │ pattern?    │              └─────────────┘
      │                      └─────────────┘                    │
      │ Y                          │ Y                           │ Y
      ▼                            ▼                             ▼
```

| Single Exponential Smoothing Method | Holt's Trend Corrected Exponential Smoothing Method | Holt-Winters Methods | Use Other Methods |

Simple Exponential Smoothing

- The <u>Simple Exponential Smoothing</u> method is used for forecasting a time series when there is no trend or seasonal pattern, but the mean (or level) of the time series y_t is slowly changing over time.

- NO TREND, No seasonality model

$$y_t = \beta_o + \varepsilon_t$$

Procedures of Simple Exponential Smoothing Method

- **Step 1**: Compute the initial estimate of the mean (or <u>level</u>) of the series at time period $t = 0$

$$\ell_0 = \bar{y} = \frac{\sum_{t=1}^{n} y_t}{n}$$

- **Step 2**: Compute the updated estimate by using the <u>smoothing equation</u>

$$\ell_T = \alpha y_{T-1} + (1 - \alpha)\ell_{T-1}$$

where y_{T-1} is the actual value , ℓ_{T-1} being the smoothing value for y_{T-1} and α is a <u>smoothing constant</u> between 0 and 1.

SIMPLE EXPONENTIAL SMOOTHING (SES)

- The equation can also be written as

$$l_t = l_{t-1} + \alpha \underbrace{(Y_{t-1} - l_{t-1})}_{\text{the forecast error}}$$

- Then, the forecast is

$$l_{t+1} = \alpha Y_t + (1 - \alpha)l_t$$

$$= l_t + \alpha(Y_t - l_t)$$

❖How is the first smoothing constant calculated?

smoothing values starts from $l_2 = Y_1$ (first value available)

❖How is this helping in forecasting values

where actual values not present?

For example, if Y6, Y7, Y8, Y9…. has to be predicted, and

only values till Y5 are given, we would be

taking the last available value i.e. Y5 in the first term

and our smoothed values would be our forecasted values.

Like $l_6 = \alpha * Y_5 + (1 - \alpha) * l_5$

   $l_7 = \alpha * Y_5 + (1 - \alpha) * l_6$

$l_8 = \alpha * Y_5 + (1 - \alpha) * l_{6 ….}$

Procedures of Simple Exponential Smoothing Method

Note that

$$\ell_T = \alpha y_{T-1} + (1 - \alpha)\ell_{T-1}$$

$$= \alpha y_{T-1} + (1 - \alpha)[\alpha y_{T-2} + (1 - \alpha)\ell_{T-2}]$$
$$= \alpha y_{T-1} + (1 - \alpha)\alpha y_{T-2} + (1 - \alpha)^2 \ell_{T-2}$$

$$= \alpha y_{T-1} + (1 - \alpha)\alpha y_{T-2} + (1 - \alpha)^2 \alpha y_{T-3} + \ldots + (1 - \alpha)^{T-2}\alpha y_1 + (1 - \alpha)^T \ell_0$$

The coefficients measuring the contributions of the observations decrease exponentially over time.

SIMPLE EXPONENTIAL SMOOTHING (SES)

- **Remarks on** α (smoothing parameter).
  - Choose α between 0 and 1.
  - If $\alpha = 1$, it becomes a naive model; if α is close to 1, more weights are put on recent values.
  - If α is close to 0, distant values are given weights comparable to recent values. Choose α close to 0 when there are big random variations in the data.
  - α is often selected as to minimize the MSE.

SIMPLE EXPONENTIAL SMOOTHING (SES)

- **Remarks on** α (smoothing parameter).
  - In empirical works, $0.05 \leq \alpha \leq 0.3$ commonly used. Values close to 1 are used rarely.
  - Numerical Minimization Process:
    - Take different α values ranging between 0 and 1.
    - Calculate 1-step-ahead forecast errors for each α.
    - Calculate MSE for each case.
    - Choose α which has the min MSE.

$$e_t = Y_t - l_t \Rightarrow \min \sum_{t=1}^{n} e_t^2 \Rightarrow \alpha$$

SIMPLE EXPONENTIAL SMOOTHING (SES)

- EXAMPLE:

Time	Y_t	$l_{t+1}(\alpha=0.10)$	$(Y_t-l_t)^2$
1	5	-	-
2	7	(0.1)5+(0.9)5=5	4
3	6	(0.1)7+(0.9)5=5.2	0.64
4	3	(0.1)6+(0.9)5.2=5.08	5.1984
5	4	(0.1)3+(0.9)5.28=5.052	1.107
		TOTAL	10.945

$$MSE = \frac{SSE}{n-1} = 2.74$$

- Calculate this for α=0.2, 0.3,…,0.9, 1 and compare the MSEs. Choose α with minimum MSE
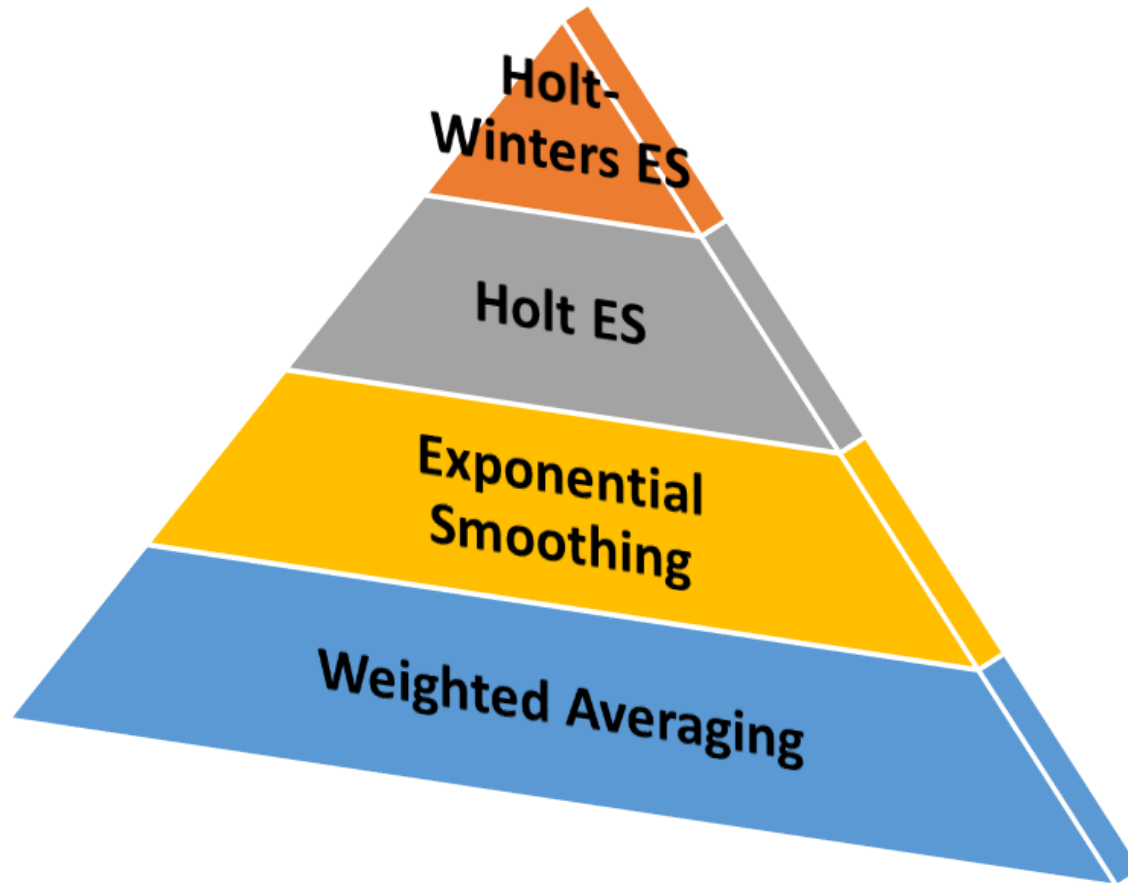
SIMPLE EXPONENTIAL SMOOTHING (SES)

- Some softwares automatically chooses the optimal α using the search method or non-linear optimization techniques.

INITIAL VALUE PROBLEM

1. Setting S_2 to Y_1 is one method of initialization.

2. Take the average of, say first 4 or 5 observations and use this as an initial value.

Smoothing Technique

DOUBLE EXPONENTIAL SMOOTHING
OR
HOLT'S EXPONENTIAL SMOOTHING

- Introduce a Trend factor to the simple exponential smoothing method
- Trend, but still **no** seasonality

$$SES + Trend = DES$$

Holt's Trend Corrected Exponential Smoothing

- If a time series is increasing or decreasing approximately at a fixed rate, then it may be described by the LINEAR TREND model

$$y_t = \beta_0 + \beta_1 t + \varepsilon_t$$

If the values of the parameters β_0 and β_1 are slowly changing over time, Holt's trend corrected exponential smoothing method can be applied to the time series observations.

<u>Note</u>: When neither β_0 nor β_1 is changing over time, regression can be used to forecast future values of y_t.

- Level (or mean) at time T: $l_T = \beta_0 + \beta_1 T$
- Growth rate (or trend): β_1

Holt's Trend Corrected Exponential Smoothing

- A smoothing approach for forecasting such a time series that employs two smoothing constants, denoted by α and γ. (Two smoothing equations are needed (Why?))

- There are two estimates ℓ_{T-1} and b_{T-1}.

  - ℓ_{T-1} is the estimate of the level of the time series constructed in time period $T–1$ (This is usually called the <u>permanent component</u>).

  - b_{T-1} is the estimate of the growth rate (β_1)(trend)of the time series constructed in time period $T–1$ (This is usually called the <u>trend component</u>).

Holt's Trend Corrected Exponential Smoothing

- Level estimate

$$\ell_T = \alpha y_T + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

- Trend estimate

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

where α = smoothing constant for the level $(0 \le \alpha \le 1)$

γ = smoothing constant for the trend $(0 \le \gamma \le 1)$

HOLT'S EXPONENTIAL SMOOTHING (Forecasting)

- Two parameters :

  α = smoothing parameter

  γ = trend coefficient

- h-step ahead forecast at time t is

$$\hat{Y}_t(h) = l_t + hb_t$$

Current level

Current slope due to trend

- Trend prediction is added in the h-step ahead forecast.

Let's use the table below to see how the calculations in Holt's Linear Method work.

Year	t	y_t	ℓ_t	b_t	$\hat{y}_t$
1989	0		17.55	4.31	
1990	1	17.55	18.41	3.62	21.86
1991	2	21.86	21.89	3.59	22.03
1992	3	23.89	24.21	3.33	25.48
1993	4	26.93	27.05	3.24	27.54

For $t = 1$ we have the following:

$\hat{y}_1 = \ell_o + h(b_o) = 17.55 + 1(4.31) = 21.86$

Updating the level (ℓ_t) and slope (b_t) we have,

$\ell_1 = \alpha y_1 + (1 - \alpha)(\ell_o + b_o) = .8(17.55) + (1 - .80)(17.55 + 4.31) = 18.41$

$b_1 = \beta^*(\ell_1 - \ell_o) + (1 - \beta^*)b_o = .2(18.41 - 17.55) + (1 - .2)(4.31) = 3.62$

For $t = 2$ we have the following:

$\hat{y}_1 = \ell_1 + h(b_1) = 18.41 + 1(3.62) = 22.03$

Updating the level (ℓ_t) and slope (b_t) we have,

$\ell_2 = \alpha y_2 + (1 - \alpha)(\ell_1 + b_1) = .8(21.86) + (1 - .8)(18.41 + 3.62) = 21.89$

$b_2 = \beta^*(\ell_2 - \ell_1) + (1 - \beta^*)b_1 = .2(21.89 - 18.41) + (1 - .2)3.62 = 3.59$

For $t = 3$ we have the following:

LEVEL+SEASONALITY+TREND
present

HOLT WINTERS EXPONENTIAL SMOOTHING METHOD

Holt-Winters Methods

- Two Holt-Winters methods are designed for time series that exhibit linear trend
  - <u>Additive Holt-Winters method</u>: used for time series with constant (additive) seasonal variations
  - <u>Multiplicative Holt-Winters method</u>: used for time series with increasing (multiplicative) seasonal variations
- Holt-Winters method is an exponential smoothing approach for handling SEASONAL data.
- The multiplicative Holt-Winters method is the better known of the two methods.

Multiplicative Holt-Winters Method

- It is generally considered to be best suited to forecasting time series that can be described by the equation:

$$y_t = (\beta_0 + \beta_1 t) \times SN_t \times IR_t$$

  - SN_t: seasonal pattern
  - IR_t: irregular component

- This method is appropriate when a time series has a linear trend with a multiplicative seasonal pattern for which the level ($\beta_0 + \beta_1 t$), growth rate (β_1), and the seasonal pattern (SN_t) may be slowly changing over time.

Multiplicative Holt-Winters Method

- Estimate of the level

$$\ell_T = \alpha(y_T / sn_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

- Estimate of the growth rate (or trend)

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

- Estimate of the seasonal factor

$$sn_T = \delta(y_T / \ell_T) + (1-\delta)sn_{T-L}$$

where α, γ, and δ are smoothing constants between 0 and 1,

L = number of seasons in a year (L = 12 for monthly data, and L = 4 for quarterly data)

HOLT-WINTER'S EXPONENTIAL SMOOTHING

Here, *(Y_t/sn_t)* captures *seasonal effects*.

   s = # of periods in the seasonal cycles

   (*s* = 4, for quarterly data)

Three parameters :

   α = smoothing parameter

   γ = trend coefficient

   δ = seasonality coefficient

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- h-step ahead forecast

$$\hat{Y}_t(h) = (l_t + hb_t)sn_{t+h-s}$$

- Seasonal factor is multiplied in the h-step ahead forecast

  α, γ and δ can be chosen as

  the value between $0.02 < \alpha, \gamma, \delta < 0.2$

or by minimizing the MSE as in SES.

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- To initialize Holt-Winter, we need at least one complete season's data to determine the initial estimates of sn_{t-s}.

- Initial value:

$$1. l_0 = \sum_{t=1}^{s} Y_t / s$$

$$2. b_0 = \frac{1}{s}\left(\frac{Y_{s+1} - Y_1}{s} + \frac{Y_{s+2} - Y_2}{s} + \cdots + \frac{Y_{s+s} - Y_s}{s} \right)$$

$$\text{or } b_0 = \left[\left\{ \sum_{t=1}^{s} Y_t / s \right\} - \left\{ \sum_{t=s+1}^{2s} Y_t / s \right\} \right] / s$$

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- For the seasonal index, say we have 6 years and 4 quarter (s=4).

STEPS TO FOLLOW

STEP 1: Compute the averages of each of 6 years.

$$A_n = \sum_{i=1}^{4} Y_i / 4, n = 1, 2, \cdots, 6 \Rightarrow \textit{The yearly averages}$$

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- **STEP 2:** Divide the observations by the appropriate yearly mean.

Year	1	2	3	4	5	6
Q_1	Y_1/A_1	Y_5/A_2	Y_9/A_3	Y_{13}/A_4	Y_{17}/A_5	Y_{21}/A_6
Q_2	Y_2/A_1	Y_6/A_2	Y_{10}/A_3	Y_{14}/A_4	Y_{18}/A_5	Y_{22}/A_6
Q_3	Y_3/A_1	Y_7/A_2	Y_{11}/A_3	Y_{15}/A_4	Y_{19}/A_5	Y_{23}/A_6
Q_4	Y_4/A_1	Y_8/A_2	Y_{12}/A_3	Y_{16}/A_4	Y_{20}/A_5	Y_{24}/A_6

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- **STEP 3:** The seasonal indices are formed by computing the average of each row such that

$$sn_1 = \left(\frac{Y_1}{A_1} + \frac{Y_5}{A_2} + \frac{Y_9}{A_3} + \frac{Y_{13}}{A_4} + \frac{Y_{17}}{A_5} + \frac{Y_{21}}{A_6} \right) / 6$$

$$sn_2 = \left(\frac{Y_2}{A_1} + \frac{Y_6}{A_2} + \frac{Y_{10}}{A_3} + \frac{Y_{14}}{A_4} + \frac{Y_{18}}{A_5} + \frac{Y_{22}}{A_6} \right) / 6$$

$$sn_3 = \left(\frac{Y_3}{A_1} + \frac{Y_7}{A_2} + \frac{Y_{11}}{A_3} + \frac{Y_{15}}{A_4} + \frac{Y_{19}}{A_5} + \frac{Y_{23}}{A_6} \right) / 6$$

$$sn_4 = \left(\frac{Y_4}{A_1} + \frac{Y_8}{A_2} + \frac{Y_{12}}{A_3} + \frac{Y_{16}}{A_4} + \frac{Y_{20}}{A_5} + \frac{Y_{24}}{A_6} \right) / 6$$

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- Note that, if a computer program selects 0 for γ and δ, this does not mean that there is no trend or seasonality.

- For Simple Exponential Smoothing, a level weight near zero implies that simple differencing of the time series may be appropriate.

- For Holt Exponential Smoothing, a level weight near zero implies that the smoothed trend is constant.

- For Winters Method and Seasonal Exponential Smoothing, a seasonal weight near one implies that a nonseasonal model may be more appropriate and a seasonal weight near zero implies that deterministic seasonal factors may be present.

Exponential smoothing with R

▣ R code:An worked out example

▣ Make sure you have "tseries" and "forecast" package.

>AirPassengers

>plot.ts(AirPassenger) #(What do you see in the data? Any seasonality? Is it constant seasonality?) If constant then Additive Holt Winters smoothing is not then multiplicative Holt Winters)

>?HoltWinters

>HoltWinters(Airpassengers, beta=FALSE,gamma=FALSE) #without trend without seasonal component

➢ Y1<-HoltWinters(Airpassengers, beta=FALSE,gamma=FALSE) $fitted # giving out predicted figures

> HoltWinters(Airpassengers, beta=FALSE,gamma=FALSE) $SSE

➢ HoltWinters(Airpassengers, seasonal=c("additive","multiplicative"))

➢ HoltWinters(Airpassengers, seasonal="additive",start.periods=1950)

➢ #Now forecasting part >library(forecast)

>z<-forecast:::forecast.HoltWinters(y1,h=10)# 10 period ahead forecast (please look at three ::: very important)

>plot(z) #Now interpret the results

>plot(z)$residuals(Also you can plot residuals)

Assignment

- 1. Retrieve the data co2 from R repository
- 2. Apply Holtwinters methodology. Which smoothing technique will you choose and why? Interpret on the values of all smoothing parameters and estimates.
- 3. Predict all observations.
- 4. Which season gives highest effect?
- 5. Plot predicted and observed data points.If you start the prediction from 10 years after time points what type of change you will notice in SSE?
- 6. Find out prediction sum of square
- 7. Next forecast next four years values
- 8. Plot forecasted values
- (For interpretation part you can look into the following website
- https://a-little-book-of-r-for-time-series.readthedocs.io/en/latest/src/timeseries.html)